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A Lagrangian relaxation approach to simultaneous strategic and tactical planning in supply chain design

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Abstract We study a multi-echelon joint inventory-location model that simultaneously determines the location of warehouses and inventory policies at the warehouses and retailers. The model is formulated as a nonlinear mixed-integer program, and is solved using a Lagrangian relaxation-based approach. The efficiency of the algorithm and benefits of integration are evaluated through a computational study.

Keywords Supply chain management · Facility location · Joint inventory-location · Lagrangian relaxation

1 Introduction

Two important considerations that arise in supply chain design are facility location and inventory management. Facility location decisions impact the firm over a significantly longer time scale than inventory management decisions; hence the former are regarded as strategic level decisions, whereas the latter are regarded as tactical level decisions. Facility location decisions and inventory management decisions are interrelated in the sense that a change in the location or number of warehouses can affect lead times and thereby affect inventory-related costs, and likewise a change in warehouse or retailer inventory policy can affect assignment decisions and thereby affect location-related costs. However, supply chain design

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has traditionally considered facility location decisions and inventory management decisions independently, leading to excess costs because the supply chain is managed sub-optimally (Cavinato 1992; Chopra and Meindl 2006; Gunasekaran et al. 2001, 2004; Silver et al. 1998; Simchi-Levi et al. 2007; Stevens 1993).

In this paper we study a model that addresses these concerns by simultaneously determining the locations of warehouses and the inventory policies at the warehouses and retailers so as to minimize system-wide costs, and we develop a Lagrangian relaxation-based heuristic approach to solving it. Much of the previous work on joint inventory-location models considered the inventory policy at warehouses, but not at retailers. However, there have been a few exceptions: You and Grossman (2010), Romeijn et al. (2007), Teo and Shu (2004), and Shu (2010) all consider the inventory policy of the retailers in their joint inventory-location models. There are, however, important differences between our work and these previous works: our inventory policy uses an Economic Order Quantity (EOQ) formulation, whereas You and Grossman formulate their inventory policy in terms of guaranteed service levels; we solve our model using a Lagrangian relaxation approach, whereas Romeijn et al. (2007), Teo and Shu (2004), and Shu (2010) solve their models by formulating them as set-covering problems.

This paper is organized as follows. In Sect. 2, we review the literature on joint inventory-location models. In Sect. 3 we describe the Multi-echelon Joint Inventory-Location (MJIL) problem. In Sect. 4 we describe a Lagrangian relaxation-based heuristic approach to solving the MJIL problem. In Sect. 5, we evaluate the effectiveness and efficiency of our algorithm through a computational study. Finally, in Sect. 6 we summarize our results and discuss future research directions.

2 Literature review

The literature on supply chain network design has traditionally considered facility location decisions and inventory management decisions independently: Amiri (2006), Daskin et al. (2005), Hindi and Pienkosz (1999), Pirkul and Jayaraman (1998), and Tsiakis et al. (2001) focus on location decisions, while Axsäter (2004), Jones and Riley (1987), Muckstadt and Roundy (1987), Svoronos and Zipkin (1991), and Wee and Yang (2004) focus on inventory management decisions.

Only recently have models that integrate these decisions been investigated. Barahona and Jensen (1998) integrate an economic order quantity (EOQ) inventory model with a location model, formulated as a large-scale integer program, and solve its linear programming relaxation using Dantzig-Wolfe decomposition. Erlebacher and Meller (2000) study an analytical joint location-inventory model that considers the fixed costs of operating warehouses, inventory holding costs at the warehouses, and transportation costs, and develop heuristic procedures to solve it. Teo et al. (2001) use an analytical modeling approach to study the impact on facility investments and inventory costs of consolidating warehouses; their model suggests that consolidation reduces costs when demands are independent and identically distributed or are independent and Poisson distributed. Miranda and Garrido (2006) propose a joint inventory-location model featuring two capacity constraints: a limit on the lot size for orders received from the warehouse, and a stochastic bound on the inventory capacity of warehouses; they solve their model using Lagrangian relaxation in conjunction with subgradient optimization.

Daskin et al. (2002) and Shen et al. (2003) study a location-inventory model with risk pooling (Eppen 1979). Daskin et al. (2002) formulate the problem as a mixed-integer nonlinear program, and, for the special case in which the ratio of the variance of demand to the mean of demand is identical for all retailers, solve the problem using a Lagrangian relaxation approach. Shen et al. (2003) formulate the problem as a set-covering problem, and solve the resulting problem using a column generation approach for two special cases: (i) the case in which the ratio of the variance of the demand to the mean of the demand is identical for all retailers, and (ii) the case in which the variance of the demand at each retailer is zero. There has been considerable effort devoted to extending the applicability of this model and to improving the associated solution procedures. Such extensions include the introduction of capacity constraints (Ozsen et al. 2008), assuming a specific functional form for the lead time (Sourirajan et al. 2007, 2009), stochastic lead times (Tanonkou et al. 2008), and relaxing the constraint that the variance-to-mean ratio of the random demands at each retailer are identical (Shu et al. 2005; Tanonkou et al. 2008; You and Grossman 2008).

There has been increasing interest in multi-echelon location-inventory models that consider inventory policy not only at warehouses, but also at retailers. You and Grossman (2010) propose a multi-echelon location-inventory model that considers location and periodic review inventory decisions simultaneously. Their formulation uses a guaranteed service level model to manage inventory at each retailer and at each warehouse, and also incorporates risk pooling at the warehouses. Romeijn et al. (2007) propose a generic modeling framework for two-echelon supply chains that considers location-specific costs, inventory costs at warehouses and retailers, and safety stock costs; by formulating their model as a set-covering problem, they obtain an efficient solution algorithm based on column generation. Teo and Shu (2004), and Shu (2010) also consider inventory policy at the retailer level in their joint inventory-location models, which are solved using a set-covering formulation.

3 Description and formulation of the problem

The multi-echelon joint inventory-location problem deals with the distribution of a single commodity from a single manufacturer to a set of retailers I (indexed by i) through a set of warehouses that can be located at various predetermined sites (indexed by j), where J is the set of possible sites. The retailers face deterministic demands and hold working inventory, representing product that has been ordered from a warehouse but has not yet been requested by end-customers. The warehouses order a single commodity from the manufacturer at regular intervals and distribute the product to the retailers. The warehouses hold working inventory representing product that has been ordered from the manufacturer but has not been yet requested by the retailers. Lateral supply among the warehouses is not allowed, that is, warehouses are supplied only by the manufacturer, and shortages are not allowed. Lead times are assumed to be negligible, although this restriction can easily be lifted to allow fixed lead times without significant changes to the model. The four major cost components in the system are:

– fixed order cost: the cost of placing an order regardless of the size of the order,
– unit-inventory cost: the cost of holding one unit of product for one unit of time,
– unit-shipping cost: the cost of shipping one unit of product between facilities, and
– fixed location cost: the cost of establishing and operating a warehouse.

The objective is to determine: (i) the number of warehouses to establish; (ii) their location; (iii) the sets of retailers that are assigned to each warehouse; and (iv) and the size and timing of orders for each facility, so as to minimize the sum of inventory, shipping, ordering, and location costs, while satisfying end-customer demand.

It has been shown for the one-warehouse multi-retailer inventory problem that if an optimal solution exists, then an optimal solution can be found that satisfies the following three properties (Roundy 1985; Schwarz 1973):

– *Zero-Inventory Ordering*: Each warehouse or retailer orders only when its inventory level reaches zero (Schwarz 1973).
– *Last-Minute Ordering*: The warehouse orders only when at least one of the retailers orders.
– *Stationarity-Between-Orders*: For each retailer, the order quantities are identical for all orders placed between two successive orders at the warehouse.

For the multi-warehouse, multi-retailer inventory problem we are considering, it is unknown if the existence of an optimal solution implies that the above three properties hold. To simplify the problem, we assume single-sourcing, in which each retailer is served by exactly one warehouse; under this assumption, for a fixed assignment of retailers to warehouses, the problem decomposes into independent one-warehouse multi-retailer problems, each of which has an optimal solution satisfying the above three properties (if an optimal solution exists at all).

In view of the above, the *Stationarity-Between-Orders* property suggests that it may be reasonable to assume that the orders placed by a given retailer have identical sizes, which suggests that a natural inventory policy to assume at the retailers is an Economic Order Quantity (EOQ) policy, which features a fixed order size for each retailer. Furthermore, by making use of the notion of *system inventory*, we can treat the demand at each warehouse as constant, which suggests that an EOQ inventory policy may be appropriate for the warehouses as well. We therefore assume an EOQ inventory policy at the warehouses and at the retailers. The EOQ inventory policy has the advantages that it is simple to implement, is widely used, and tends to work well in practice.

The problem we are considering is a generalization of the one-warehouse multi-retailer inventory problem, in that we are also considering multiple warehouses and the assignment of retailers to warehouses. Given that even the one-warehouse multi-retailer inventory problem is difficult to solve to optimality (Roundy 1985), we must make some simplifying assumptions. We therefore assume power-of-two inventory policies, which have the desirable property of being 98% effective for the one-warehouse multi-retailer problem (Roundy 1985), meaning that the ratio of the average cost of the best policy to the average cost of the power-of-two policy is at least 0.98. We will show later that a power-of-two inventory policy is also 98% effective for our problem, for a fixed assignment of retailers to warehouses.

A power-of-two inventory policy is one in which the ratio of the time between orders at the warehouse to the time between orders at each retailer it serves is a power of two. Let $\widehat{T}_j$ be the time between orders placed by warehouse j to the plant, and let T_{ij} be the time between orders placed by retailer i to warehouse j (assuming that retailer i is served by warehouse j). Then, assuming a power-of-two inventory policy,

$$\widehat{T}_j / T_{ij} = 2^{N_{ij}}$$

where $N_{ij} \in \mathbb{Z}$ for all i, j. We assume that $\widehat{T}_j$ and T_{ij} are given in days.

Consider now the inventory problem faced by a single warehouse j and the retailers it serves. Let $\Im_j$ be the set of retailers served by warehouse j, and let $\hat{h}_j$, $\hat{o}_j$ be respectively the unit-inventory holding cost per unit time at warehouse j, and the fixed cost per order at warehouse j. Let h_i, o_i be respectively the unit-inventory holding cost per unit time at retailer i, and the fixed cost per order at retailer i. Since the number of orders placed by

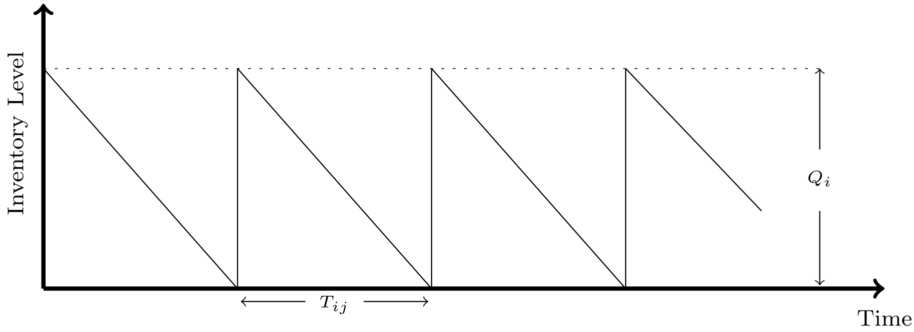

Fig. 1 An inventory-time plot for retailer i

warehouse j to the plant per base planning period is $t_b/\widehat{T}_j$, and the cost per order is $\hat{o}_j$, the ordering cost per base planning period at warehouse j is $\hat{o}_j t_b/\widehat{T}_j$. Let $\hat{C}_j$ denote the total inventory and ordering costs over the base planning period for the system composed of warehouse j and the retailers it serves. Then $\hat{C}_j$, which is a function of $\widehat{T}_j$ and $\{T_{ij} : i \in \Im_j\}$, has the form

$$\hat{C}_j(\widehat{T}_j, \{T_{ij} : i \in \Im_j\}) = \frac{\hat{o}_j}{\widehat{T}_j} t_b + \sum_{i \in \Im_j} C_{ij}(\widehat{T}_j, T_{ij}) \tag{1}$$

where C_{ij} represents the inventory and ordering costs imputable to retailer i. We next derive an explicit form for C_{ij}. Consider $\widehat{T}_j$ as fixed. Then the retailers i fall into two groups: those for which $T_{ij} \le \widehat{T}_j$, and those for which $T_{ij} > \widehat{T}_j$.

Case 1: $T_{ij} > \widehat{T}_j$ In this case, the warehouse places an order simultaneously with every order placed by retailer i, hence the warehouse will not carry an inventory of goods to be shipped to retailer i. It follows that the inventory cost at the warehouse for goods to be shipped to retailer i is zero. We next determine the inventory and ordering costs at retailer i. Let Q_i be the order quantity for retailer i and T_{ij} be the length of an order cycle for retailer i (assumed to have units of days). Under our assumptions, the demand at retailer i has the characteristic sawtooth pattern shown in Fig. 1. The average inventory level at retailer i is the area under the time-inventory plot for one order cycle, or $\frac{1}{2} Q_i T_{ij}$, divided by the order cycle length T_{ij}, or $\frac{1}{2} Q_i$. Because the daily demand d_i is constant, it follows from our assumptions that

$$Q_i = d_i T_{ij} \tag{2}$$

so that the average inventory level at retailer i is

$$\mathrm{AvgInv}_i = \frac{1}{2} \underbrace{Q_i}_{d_i T_{ij}} = \frac{1}{2} d_i T_{ij}. \tag{3}$$

Then the total inventory holding cost at retailer i over the base planning period is the holding cost per unit per day, h_i, multiplied by the average daily inventory level at re-

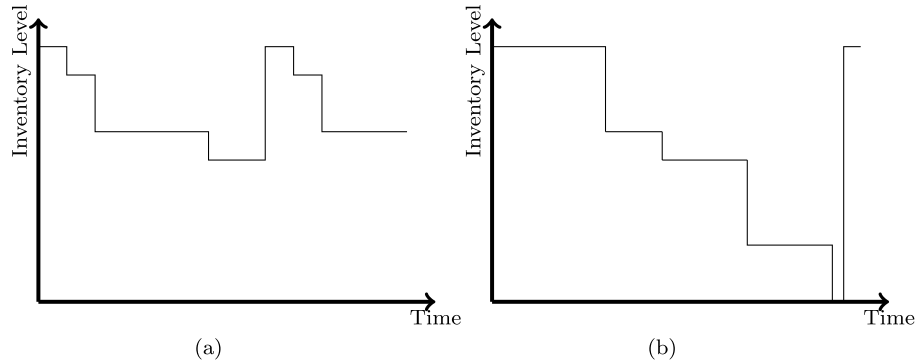

Fig. 2 Two possible inventory-time plots for warehouse j

tailer i, multiplied by the number of days per base planning period, t_b, so that

$$\begin{aligned}\text{TotalHoldingCost}_i &= h_i \cdot \underbrace{\text{AvgInv}_i}_{\frac{1}{2}d_i T_{ij}} \cdot \underbrace{\text{number of days per base planning period}}_{t_b} \\ &= \frac{1}{2} h_i d_i T_{ij} t_b. \end{aligned} \tag{4}$$

To (4) we must add the ordering costs at the retailer. Since the number of orders placed by retailer i to warehouse j is t_b/T_{ij}, and the cost per order is o_i, the cost of orders placed by retailer i over the base planning period is $o_i t_b/T_{ij}$. It follows that for this case

$$C_{ij}(\widehat{T}_j, T_{ij}) = \frac{o_i}{T_{ij}} t_b + \frac{1}{2} h_i d_i T_{ij} t_b. \tag{5}$$

Case 2: $T_{ij} \le \widehat{T}_j$ For this case the inventory holding cost at the warehouse will be nonzero. In general, an inventory vs. time plot for the warehouse can have an arbitrary shape, as shown in Fig. 2, making it difficult to determine the average inventory level. To overcome this problem, Roundy (1985) used the echelon method of computing holding costs (Clark and Scarf 1960), and we shall do the same. Define the *system inventory* at a retailer i that is served by warehouse j as the inventory at retailer i plus the inventory at the warehouse that is destined for retailer i. Then the system inventory has a sawtooth pattern with an order interval of $\widehat{T}_j$ (Roundy 1985), as shown in Fig. 3. We can therefore calculate the contribution to the holding costs due to the system inventory using the same method that was used in Case 1 to find the total inventory holding cost at retailer i over the base planning period, except that $\widehat{T}_j$ is used in place of T_{ij}, and the per-unit holding cost at the warehouse, $\hat{h}_j$, is used in place of h_i, thus obtaining

$$\frac{1}{2} \hat{h}_j d_i \widehat{T}_j t_b. \tag{6}$$

To (6) we must add the inventory holding cost at the retailer, which is calculated as in Case 1, except that instead of using h_i, we use the per-unit echelon holding cost at retailer i, $h_i - \hat{h}_j$, thus obtaining

$$\frac{1}{2}(h_i - \hat{h}_j) d_i T_{ij} t_b. \tag{7}$$

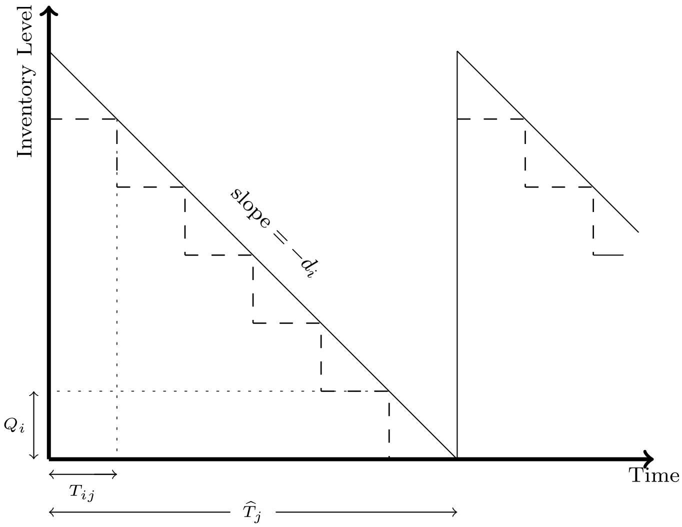

Fig. 3 System inventory for retailer i served by warehouse j

We also must include the ordering cost at the retailer, which, by the same calculation used in Case 1, is $o_i t_b / T_{ij}$. Adding these costs, we find that

$$\begin{aligned} C_{ij}(\widehat{T}_j, T_{ij}) &= \frac{o_i}{T_{ij}} t_b + \frac{1}{2}(h_i - \hat{h}_j) d_i T_{ij} t_b + \frac{1}{2} \hat{h}_j d_i \widehat{T}_j t_b \\ &= \frac{o_i}{T_{ij}} t_b + \frac{1}{2}(h_i - \hat{h}_j) d_i T_{ij} t_b + \frac{1}{2} \hat{h}_j d_i \max\{\widehat{T}_j, T_{ij}\} t_b. \end{aligned} \tag{8}$$

The expressions for C_{ij} in Case 1 (5) and Case 2 (8) can be unified as

$$C_{ij}(\widehat{T}_j, T_{ij}) = \frac{o_i}{T_{ij}} t_b + \frac{1}{2}(h_i - \hat{h}_j) d_i T_{ij} t_b + \frac{1}{2} \hat{h}_j d_i \max\{\widehat{T}_j, T_{ij}\} t_b. \tag{9}$$

Substituting (9) into (1), we find that the total inventory and ordering costs for warehouse j and the retailers it serves over the base planning period are

$$\begin{aligned} \hat{C}_j(\widehat{T}_j, \{T_{ij} : i \in \Im_j\}) &= \frac{\hat{o}_j}{\widehat{T}_j} t_b + \sum_{i \in \Im_j} C_{ij}(\widehat{T}_j, T_{ij}) \\ &= \frac{\hat{o}_j}{\widehat{T}_j} t_b + \sum_{i \in \Im_j} \left(\frac{o_i}{T_{ij}} t_b + \frac{1}{2}(h_i - \hat{h}_j) d_i T_{ij} t_b + \frac{1}{2} \hat{h}_j d_i \max\{\widehat{T}_j, T_{ij}\} t_b \right). \end{aligned} \tag{10}$$

To specify the open warehouses and the assignments of retailers to warehouses, we introduce decision variables X_j and Y_{ij} as follows:

$$X_j = \begin{cases} 1 & \text{if a warehouse is opened at candidate location } j \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{ij} = \begin{cases} 1 & \text{if retailer } i \text{ is served by the warehouse at location } j \\ 0 & \text{otherwise.} \end{cases}$$

Using these variables in (10) and summing over all warehouses j, we obtain

$$\sum_{j\in J} \frac{\hat{o}_j t_b}{\widehat{T}_j} X_j + \sum_{j\in J}\sum_{i\in I}\left(\frac{o_i t_b}{T_{ij}} + \frac{1}{2}(h_i - \hat{h}_j) d_i t_b T_{ij} + \frac{1}{2}\hat{h}_j d_i t_b \max\{\widehat{T}_j, T_{ij}\}\right) Y_{ij}. \tag{11}$$

To (11), which only takes account of inventory and ordering costs, we must add the location and shipping costs. Let f_j be the fixed cost of opening and operating warehouse j, let s_{ij} be the per-unit shipping cost from warehouse j to retailer i, and let $\hat{s}_j$ be the per-unit shipping cost from the plant to warehouse j. Adding the costs of opening and operating warehouses and shipping costs to (11), we obtain:

$$\begin{aligned} &\sum_{j\in J}\sum_{i\in I}\left(\frac{o_i t_b}{T_{ij}} + \frac{1}{2}(h_i - \hat{h}_j) d_i t_b T_{ij} + \frac{1}{2}\hat{h}_j d_i t_b \max\{\widehat{T}_j, T_{ij}\}\right) Y_{ij} \\ &\quad + \sum_{j\in J} \frac{\hat{o}_j t_b}{\widehat{T}_j} X_j + \sum_{j\in J} f_j X_j + \sum_{j\in J} \hat{s}_j \left(\sum_{i\in I} d_i t_b Y_{ij}\right) + \sum_{j\in J}\sum_{i\in I} d_i t_b s_{ij} Y_{ij} \end{aligned} \tag{12}$$

where we have used the fact that the demand seen by retailer i over the base planning period is $d_i t_b$. Rearranging terms in (12), and introducing a weighting factor β_{trn} for transportation costs and a weighting factor β_{inv} for inventory and ordering costs gives

$$\begin{aligned} &\sum_{j\in J}\sum_{i\in I}\left(\beta_{\text{inv}}\frac{o_i t_b}{T_{ij}} + \frac{1}{2}\beta_{\text{inv}}(h_i - \hat{h}_j) d_i t_b T_{ij} + \frac{1}{2}\beta_{\text{inv}}\hat{h}_j d_i t_b \max\{\widehat{T}_j, T_{ij}\}\right) Y_{ij} \\ &\quad + \sum_{j\in J}\left(f_j + \beta_{\text{inv}}\frac{\hat{o}_j t_b}{\widehat{T}_j}\right) X_j + \sum_{j\in J}\sum_{i\in I} \beta_{\text{trn}} d_i t_b (\hat{s}_j + s_{ij}) Y_{ij}. \end{aligned} \tag{13}$$

For convenience we define

$$\begin{aligned} k_i &= \beta_{\text{inv}} o_i t_b \\ \hat{k}_j &= \beta_{\text{inv}} \hat{o}_j t_b \\ b_{ij} &= \beta_{\text{trn}} (\hat{s}_j + s_{ij}) d_i t_b \\ c_{ij} &= \frac{1}{2}\beta_{\text{inv}}(h_i - \hat{h}_j) d_i t_b \\ e_{ij} &= \frac{1}{2}\beta_{\text{inv}} \hat{h}_j d_i t_b. \end{aligned}$$

Then the total inventory, ordering, location, and shipping costs can be expressed as

$$\sum_{j\in J}\left(f_j + \frac{\hat{k}_j}{\widehat{T}_j}\right) X_j + \sum_{j\in J}\sum_{i\in I}\left(\frac{k_i}{T_{ij}} + b_{ij} + c_{ij} T_{ij} + e_{ij} \max\{\widehat{T}_j, T_{ij}\}\right) Y_{ij}. \tag{14}$$

We can now formulate the Multi-echelon Joint Inventory-Location (MJIL) problem as

$$\min_{X,Y,\widehat{T},T} \quad \sum_{j\in J}\left(f_j + \frac{\hat{k}_j}{\widehat{T}_j}\right)X_j + \sum_{j\in J}\sum_{i\in I}\left(\frac{k_i}{T_{ij}} + b_{ij} + c_{ij}T_{ij} + e_{ij}\max\{\widehat{T}_j, T_{ij}\}\right)Y_{ij} \tag{15}$$

$$\text{s.t.} \quad \sum_{j\in J} Y_{ij} = 1 \qquad \forall i \in I \tag{16}$$

$$Y_{ij} \le X_j \qquad \forall i \in I, \forall j \in J \tag{17}$$

$$Y_{ij} \in \{0,1\} \qquad \forall i \in I, \forall j \in J \tag{18}$$

$$X_j \in \{0,1\} \qquad \forall j \in J \tag{19}$$

$$\widehat{T}_j / T_{ij} = 2^{N_{ij}} \qquad \forall i \in I, \forall j \in J \tag{20}$$

$$N_{ij} \in \mathbb{Z} \qquad \forall i \in I, \forall j \in J \tag{21}$$

$$\widehat{T}_j > 0 \qquad \forall j \in J \tag{22}$$

$$T_{ij} > 0 \qquad \forall i \in I, \forall j \in J. \tag{23}$$

Constraints (16) require that each retailer be assigned to exactly one warehouse. Constraints (17) prevent retailers from being assigned to warehouses that are closed. Constraints (18) and (19) require that Y_{ij} and X_j be binary. Finally, Constraints (20) through (23) require a power-of-two inventory policy at the retailers and the warehouses.

Next we reformulate (15)–(23) as a nonlinear uncapacitated facility location problem by first optimizing over T and $\widehat{T}$ for any given X and Y satisfying (16)–(19), and then optimizing over X and Y. We obtain:

$$\min_{X,Y} \quad \sum_{j\in J}\left(f_j X_j + \sum_{i\in I} b_{ij}Y_{ij} + Z_j^*(\Im_j, X_j)\right) \tag{24}$$

$$\text{s.t.} \quad \sum_{j\in J} Y_{ij} = 1 \qquad \forall i \in I, \forall j \in J \tag{25}$$

$$Y_{ij} \le X_j \qquad \forall i \in I, \forall j \in J \tag{26}$$

$$Y_{ij} \in \{0,1\} \qquad \forall i \in I, \forall j \in J \tag{27}$$

$$X_j \in \{0,1\} \qquad \forall j \in J \tag{28}$$

where

$$\Im_j = \{i \in I | Y_{ij} = 1\}$$

and where

$$Z_j^*(\Im_j, X_j) = \min_{\widehat{T}_j, T_{ij}} \left\{ \begin{array}{l} \frac{\hat{k}_j}{\widehat{T}_j} X_j + \sum_{i\in\Im_j}\left(\frac{k_i}{T_{ij}} + c_{ij}T_{ij} + e_{ij}\max(T_{ij}, \widehat{T}_j)\right) \\ \text{s.t. } \widehat{T}_j/T_{ij} = 2^{N_{ij}} \text{ where } N_{ij} \in \mathbb{Z},\ T_{ij} > 0, \text{ and } \widehat{T}_j > 0 \end{array} \right\}.$$

4 Solving the MJIL problem using Lagrangian relaxation

The MJIL problem is a generalization of the uncapacitated fixed-charge location problem, which is known to be NP-hard (Daskin 1995); it follows that the MJIL problem is also NP-hard. A number of algorithms have shown success in solving such problems, including Lagrangian relaxation (Chen and Chu 2003; Eskigun et al. 2005; Fisher 1981, 1985; Jayaraman and Pirkul 2001; Min et al. 2005; Pirkul and Jayaraman 1998; Putz 2007), column generation (Romeijn et al. 2007), and various heuristic methods such as genetic algorithms (Chan et al. 2005), simulated annealing (Jayaraman and Ross 2003), and tabu search (Shiguemoto and Armentano 2010); here we develop a Lagrangian relaxation approach to solving the MJIL problem.

The first step in developing this approach is to consider the Lagrangian dual problem obtained by relaxing Constraints (25):

$$\max_{\lambda\geq 0}\min_{X,Y} \quad \sum_{j\in J}\Bigl(f_j X_j + \sum_{i\in I} b_{ij}Y_{ij} + Z_j^*(\Im_j, X_j)\Bigr) + \sum_{i\in I}\lambda_i\Bigl(1-\sum_{j\in J}Y_{ij}\Bigr)$$

$$= \max_{\lambda\geq 0}\sum_{j\in J}\min_{X_j,\{Y_{ij}:i\in I\}}\Bigl(f_j X_j + \sum_{i\in I}(b_{ij}-\lambda_i)Y_{ij} + Z_j^*(\Im_j, X_j)\Bigr) + \sum_{i\in I}\lambda_i$$

$$\text{s.t.}\quad Y_{ij}\leq X_j \qquad \forall i\in I, \forall j\in J \tag{29}$$

$$Y_{ij}\in\{0,1\} \quad \forall i\in I, \forall j\in J \tag{30}$$

$$X_j\in\{0,1\} \quad \forall j\in J \tag{31}$$

$$= \max_{\lambda\geq 0}\sum_{j\in J} S_j^*(\lambda) + \sum_{i\in I}\lambda_i \tag{32}$$

where

$$S_j^*(\lambda) = \min_{X_j,\{Y_{ij}:i\in I\}}\left\{\begin{array}{l} f_j X_j + \sum_{i\in I}(b_{ij}-\lambda_i)Y_{ij} + Z_j^*(\Im_j, X_j) \\ \text{s.t. } Y_{ij}\leq X_j \ \forall i\in I,\ X_j\in\{0,1\}, \text{ and } Y_{ij}\in\{0,1\}\ \forall i\in I\end{array}\right\}. \tag{33}$$

Thus we see that the Lagrangian relaxation problem decomposes into $|J|$ subproblems, one for each warehouse $j\in J$. The subproblem corresponding to warehouse j is denoted $S_j(\lambda)$, and the value of the objective function of this subproblem at optimality is denoted $S_j^*(\lambda)$. The strategy we will use to solve the subproblems $S_j(\lambda)$ is to alternate between optimizing over X_j and $\{Y_{ij}: i\in I\}$ for fixed $\widehat{T}_j$ and $\{T_{ij}: i\in I\}$, and optimizing over $\widehat{T}_j$ and $\{T_{ij}: i\in \Im_j\}$ for fixed X_j and $\{Y_{ij}: i\in I\}$ until a convergence criterion is satisfied. For a fixed assignment of retailers to warehouses, that is, for fixed X_j and Y_{ij}, $Z_j^*(\Im_j, X_j)$ is the optimal value of a power-of-two multi-echelon inventory problem for warehouse j, and is thus an instance of the problem considered by Roundy (1985). Roundy developed an algorithm to solve this problem which relaxes the power-of-two constraints; the solution to this relaxed problem, when appropriately rounded, gives a nearly optimal solution to the problem with the power-to-two constraints in place. More specifically, Roundy showed that for one warehouse serving multiple retailers, this approach to generating a power-of-two inventory policy is at least 98% effective (Roundy 1985), meaning that if $Z'_j(\Im_j, X_j)$ is the contribution to the inventory costs, on average, of a single warehouse j and the retailers it serves for the power-of-two inventory policy found using the above algorithm, and if $\tilde{Z}_j(\Im_j, X_j)$ is the infimum of the average cost over all policies at warehouse j, then

$$100\times\frac{\tilde{Z}_j(\Im_j, X_j)}{Z'_j(\Im_j, X_j)}\geq 98. \tag{34}$$

However, the inventory term in our objective function (24) has the form $\sum_{j\in J} Z_j(\Im_j, X_j)$, so we need to bound

$$\frac{\sum_{j\in J} \tilde{Z}_j(\Im_j, X_j)}{\sum_{j\in J} Z'_j(\Im_j, X_j)}. \tag{35}$$

Such a bound can be obtained as follows. From (34) it follows that

$$100 \times \tilde{Z}_j(\Im_j, X_j) \geq 98 \times Z'_j(\Im_j, X_j). \tag{36}$$

Hence, summing over j, we obtain

$$100 \times \sum_{j\in J} \tilde{Z}_j(\Im_j, X_j) \geq 98 \times \sum_{j\in J} Z'_j(\Im_j, X_j) \tag{37}$$

or

$$100 \times \frac{\sum_{j\in J} \tilde{Z}_j(\Im_j, X_j)}{\sum_{j\in J} Z'_j(\Im_j, X_j)} \geq 98. \tag{38}$$

If follows that the overall power-of-two inventory policy considering all warehouses j is 98% efficient for a fixed assignment of retailers to warehouses. As Roundy's algorithm provides a means of obtaining a near-optimal power-of-two inventory policy for fixed X_j and $\{Y_{ij} : i \in I\}$, we use this method to obtain $\widehat{T}_j$ and $\{T_{ij} : i \in \Im_j\}$ for fixed X_j and $\{Y_{ij} : i \in I\}$.

We now turn to the optimization over X_j and $\{Y_{ij} : i \in I\}$ for fixed $\widehat{T}_j$ and $\{T_{ij} : i \in I\}$ in subproblem $S_j(\lambda)$ (33). For fixed $\widehat{T}_j$ and $\{T_{ij} : i \in I\}$, the coefficients of X_j and $\{Y_{ij} : i \in I\}$ in subproblem $S_j(\lambda)$ become constant, so that the optimal values of X_j and $\{Y_{ij} : i \in I\}$ can be found by the following algorithm:

1. If the coefficient of Y_{ij} is positive and Y_{ij} is 1, set $Y_{ij} = 0$.
2. Fixing $\{Y_{ij} : i \in I\}$, and keeping $\widehat{T}_j$ and $\{T_{ij} : i \in I\}$ fixed, set $X_j = 0$ or 1 according to whichever value minimizes the objective function (33). If a zero value results for X_j, go back and set $Y_{ij} = 0$ for $i \in I$.

Solving the subproblems $S_j(\lambda)$ (33) and substituting the result back into the objective function (32) gives a lower bound on the objective function value for the MJIL problem, provided the subproblems $S_j(\lambda)$ are solved to optimality. If any of the subproblems are not solved to optimality, then the "lower bound" on the optimal objective function value of the original problem that results from a particular value of λ may not be a lower bound at all. Since in our case the subproblems $S_j(\lambda)$ are nonlinear and nonconvex in the decision variables, they are difficult to solve to optimality. Nevertheless, if the algorithm that carries out the minimizations indicated by the subproblems $S_j(\lambda)$ is sufficiently effective, the overall process may still function as a useful heuristic for obtaining a good solution to the original problem, but with no guarantee that the "lower bound" is a true lower bound.

An upper bound on the MJIL problem, which is useful for the optimization over the λ_i's, is provided by any feasible solution. Our approach to constructing a feasible solution uses the solution of (32) as a starting point. Retailers i that have not been assigned to any warehouse are assigned to the warehouse j' having the smallest coefficient of $Y_{ij'}$ in (15) over all the warehouses that are open, while retailers i that have been assigned to multiple warehouses j belonging to a set J' are assigned to the warehouse $j' \in J'$ having the smallest coefficient of $Y_{ij'}$ in (15). This procedure produces a solution $(\tilde{X}, \tilde{Y})$ that satisfies (25)–(28). Further details are provided in procedure **CalculateFeasibleSolution** (Algorithm 1).

Data: Lower bound solution $X, Y, T, \widehat{T}$; matrices b, c, e; vectors $\hat{k}, k$
Result: Feasible solution $\tilde{X}, \tilde{Y}$ to MJIL problem

begin
  $\tilde{Y} \longleftarrow Y$
  for $i \in I$ **do** `// For each retailer i`
    $nwarehouses \longleftarrow 0$
    for $j \in J$ **do** `// For each warehouse j`
      if $Y_{ij} = 1$ **then**
        $nwarehouses \longleftarrow nwarehouses + 1$
      end
    end
    switch *nwarehouses* **do**
      case *0*
        Of the warehouses $j \in J$ such that $X_j = 1$, find one, call it j', with the smallest coefficient of $Y_{ij'}$ in the objective function for the original problem at fixed $\widehat{T}$ and T
        $\tilde{Y}_{ij'} \longleftarrow 1$
        $\tilde{Y}_{ij} \longleftarrow 0$ for $j \neq j'$
      end
      case *1*
      end
      otherwise
        Of the warehouses $j \in J$ such that $Y_{ij} = 1$, find one, call it j', with the smallest coefficient of $Y_{ij'}$ in the objective function for the original problem at fixed $\widehat{T}$ and T
        $\tilde{Y}_{ij'} \longleftarrow 1$
        $\tilde{Y}_{ij} \longleftarrow 0$ for $j \neq j'$
      end
    end
  end
  for $j \in J$ **do** `// For each warehouse j`
    if $\tilde{Y}_{ij} = 1$ *for at least one retailer i* **then**
      $\tilde{X}_j \longleftarrow 1$
    else
      $\tilde{X}_j \longleftarrow 0$
    end
  end
  return $(\tilde{X}, \tilde{Y})$
end

Algorithm 1: CalculateFeasibleSolution

The overall Lagrangian relaxation procedure consists of an outer loop and inner loop. The outer loop maximizes over the λ_i's using a subgradient optimization procedure (see Putz 2007), while the inner loop minimizes over X, Y, T, and $\widehat{T}$. The subgradient optimization uses an "agility" parameter π, which is initially set to 2, and is reduced by dividing by an "agility reduction factor" (denoted α) every time N_π iterations are performed without improvement in the solution. We experimented with several different values of α and N_π, and chose values for these parameters that gave the best results; in particular, we set $\alpha = 1.2$ and $N_\pi = 100$. Since the lower bound obtained is not necessarily a true lower bound, the algorithm includes a test to ensure that the best upper bound at a given point in the optimization is greater than the best lower bound to this point; if not, the best lower bound solution is discarded. Details of the Lagrangian relaxation are provided in procedure **SolveMJIL** (Algorithm 2).

```
Data: Matrices b, c, e; vectors k̂, k
Result: Solution (X, Y, T, T̂) to MJIL problem
begin
    π ⟵ 2
    IterationsWithoutImprovement ⟵ 0
    repeat                                                  // Loop to optimize over λ
        for j ∈ J do                                        // For each warehouse j
            Y_ij ⟵ 1 for each i ∈ I
            X_j ⟵ 1
            [{T_ij : i ∈ I}, T̂_j] ⟵ SolveRoundyProblem(X_j, {Y_ij : i ∈ I})
            repeat                              // Loop to optimize over X, Y, T, and T̂
                if Y_ij = 1 for at least one retailer i ∈ I then
                    for i ∈ I do                            // For each retailer i
                        if Y_ij = 1 and the coefficient of Y_ij in the objection function to
                        subproblem S_j(λ) is positive then
                            Y_ij ⟵ 0
                        end
                    end
                    if Objective function for subproblem S_j(λ) is negative then
                        X_j ⟵ 1
                    else
                        X_j ⟵ 0
                        Y_ij ⟵ 0 for all i ∈ I
                    end
                    [{T_ij : i ∈ I}, T̂_j] ⟵ SolveRoundyProblem(X_j, {Y_ij : i ∈ I})
                end
            until No change in X_j or {Y_ij : i ∈ I}
        end
        CurrentLowerBound ⟵ CalculateObjectiveFunctionValue(X, Y, T, T̂)
        [X̃, Ỹ] ⟵ CalculateFeasibleSolution(X, Y, T, T̂)
        for j ∈ J do                                        // For each warehouse j
            [{T̃_ij : i ∈ ℑ_j}, T̂̃_j] ⟵ SolveRoundyProblem(X̃_j, {Ỹ_ij : i ∈ ℑ_j})
        end
        CurrentUpperBound ⟵ CalculateObjectiveFunctionValue(X̃, Ỹ, T̃, T̂̃)
        δ_i ⟵ 1 − ∑_{j∈J} Y_ij        for i ∈ I
        Δ ⟵ π(|BestUpperBound − CurrentLowerBound|) / ∑_{i∈I} δ_i²
        λ_i ⟵ max[0, λ_i + Δ · δ_i]
        if CurrentUpperBound < BestUpperBound then
            BestUpperBound ⟵ CurrentUpperBound
            X* ⟵ X̃, Y* ⟵ Ỹ, T* ⟵ T̃, T̂* ⟵ T̂̃
        end
        if BestUpperBound < BestLowerBound then
            BestLowerBound ⟵ −10^99
        end
        if CurrentLowerBound > BestLowerBound and
          CurrentLowerBound ≤ BestUpperBound then
            BestLowerBound ⟵ CurrentLowerBound
        else
            IterationsWithoutImprovement ⟵ IterationsWithoutImprovement + 1
            if IterationsWithoutImprovement > 100 then
                IterationsWithoutImprovement ⟵ 0
                π ⟵ π/1.2
            end
        end
    until (|BestUpperBound − BestLowerBound| < .000001) or (π < .005)
    return (X*, Y*, T*, T̂*)
end
```

Algorithm 2: SolveMJIL

5 Computational study

We tested our algorithm on the following data sets, derived from Daskin (1995):

- a 49-node data set (48 continental US state capitals plus Washington, DC; 1990 census data);
- an 88-node data set (49-node dataset plus the 50 largest US cities minus duplicates; 1990 census data); and
- a 150-node data set (150 largest US cities; 1990 census data).

The Lagrangian relaxation algorithm described in Sect. 4 was implemented in C#, and computer experiments were run on a 2.8 GHz dual core processor with 8 GB of RAM.

In order to determine the value of integrating location and inventory decisions, we generated 36 scenarios for each data set by varying the values of the transportation cost weighting factor β_{trn} and the inventory cost weighting factor β_{inv} such that

$$\beta_{\text{trn}} \in \{0.001, 0.01, 0.1, 1, 10, 100\} \quad \text{and} \quad \beta_{\text{inv}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}.$$

These values allow the benefits of integration to be evaluated for different relative weightings of transportation costs and inventory costs.

We computed the Value of Integration (VOI), which measures the benefit obtained from integrating location, transportation, and inventory decisions, as follows:

$$\text{Value of Integration} = 100 \times \frac{(\text{Best UB for Sequential Opt.}) - (\text{Best UB for Joint Opt.})}{\text{Best UB for Joint Opt.}}.$$

In the above expression, "Best UB for Joint Opt." refers to the best upper bound obtained by integrating location, transportation, and inventory decisions, while "Best UB for Sequential Opt." refers to the best upper bound obtained by first solving a transportation-location problem to obtain an assignment of retailers to warehouses, and then solving an inventory problem at each warehouse based on these assignments.

Table 1 provides results for a small problem with 11 retailers and 5 warehouses that is derived from the 49-city data set for which the optimal solution can be calculated by exhaustive search; this table shows that the algorithm found an optimal solution in 36 out of 36 scenarios. On this problem, there was only one positive VOI, with all other VOI being zero, indicating that over much of the $(\beta_{\text{trn}}, \beta_{\text{inv}})$ parameter space, there is little benefit to integration. The positive VOI occurred at $(\beta_{\text{trn}}, \beta_{\text{inv}}) = (0.1, 100)$, indicating a clear benefit to integration in this region of the parameter space.

Tables 2–4 provide results for the 49-city, 88-city, and 150-city data sets, in which every city represents a retailer location as well as a candidate warehouse location. Again, most of the VOI were zero, indicating that there is little benefit to integration over much of the $(\beta_{\text{trn}}, \beta_{\text{inv}})$ parameter space. However, there were some positive VOI: for the 49-city data set, positive VOI occurred at $(\beta_{\text{trn}}, \beta_{\text{inv}}) = (0.1, 10), (0.1, 100), (1, 10), (1, 100)$; for the 88-city data set, positive VOI occurred at $(\beta_{\text{trn}}, \beta_{\text{inv}}) = (0.01, 10), (0.01, 100), (.1, 1), (.1, 10), (.1, 100)$; and for the 150-city data set, positive VOI occurred at $(\beta_{\text{trn}}, \beta_{\text{inv}}) = (0.1, 100), (1, 100)$. These positive VOI reveal a clear benefit to integration over the corresponding regions of the parameter space.

Tables 1–4 also reveal that in some cases the VOI was negative, indicating that sequential (non-integrated) optimization gave a better solution than joint (integrated) optimization. One of the reasons that this can occur is that the joint optimization approach uses Lagrangian relaxation to solve a combined transportation-location-inventory problem. In this case the

Table 1 Results for 11 retailers and 5 candidate warehouse locations chosen from the 49-city data set (retailers in cities $\{1-11\}$, candidate warehouse locations in cities $\{4, 11, 24, 35, 47\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$, $\beta_{\text{inv}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 49-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 49-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI	Optimal value
1	0.001	0.001	1	0.031	216368.057	2.910E−11	0.0000	216368.057
2	0.001	0.010	1	0.016	217119.219	2.910E−11	0.0000	217119.219
3	0.001	0.100	1	0.016	224630.836	8.731E−11	0.0000	224630.836
4	0.001	1	1	0.016	299747.009	0.000E+00	0.0000	299747.009
5	0.001	10	1	0.031	1050908.738	0.000E+00	0.0000	1050908.738
6	0.001	100	1	0.078	8562526.023	1.863E−09	0.0000	8562526.023
7	0.010	0.001	1	0.062	538565.345	0.000E+00	0.0000	538565.345
8	0.010	0.010	1	0.062	539316.507	0.000E+00	0.0000	539316.507
9	0.010	0.100	1	0.640	546828.124	0.000E+00	0.0000	546828.124
10	0.010	1	1	0.047	621944.297	0.000E+00	0.0000	621944.297
11	0.010	10	1	0.078	1373106.025	2.328E−10	0.0000	1373106.025
12	0.010	100	1	0.031	8884723.311	1.863E−09	0.0000	8884723.311
13	0.100	0.001	2	0.000	3333735.921	0.000E+00	0.0000	3333735.921
14	0.100	0.010	2	0.031	3334522.496	0.000E+00	0.0000	3334522.496
15	0.100	0.100	2	0.031	3342388.244	0.000E+00	0.0000	3342388.244
16	0.100	1	2	0.016	3421045.732	0.000E+00	0.0000	3421045.732
17	0.100	10	2	0.031	4207620.605	0.000E+00	0.0000	4207620.605
18	0.100	100	1	0.016	11904627.672	1.863E−09	1.4174	11904627.672
19	1	0.001	3	0.000	27890684.715	0.000E+00	0.0000	27890684.715
20	1	0.010	3	0.047	27891499.912	3.725E−09	0.0000	27891499.912
21	1	0.100	3	0.016	27899651.890	0.000E+00	0.0000	27899651.890
22	1	1	3	0.000	27981171.665	0.000E+00	0.0000	27981171.665
23	1	10	3	0.016	28796369.420	0.000E+00	0.0000	28796369.420
24	1	100	3	0.000	36948346.963	0.000E+00	0.0000	36948346.963
25	10	0.001	4	0.000	271799222.557	0.000E+00	0.0000	271799222.557
26	10	0.010	4	0.000	271800052.834	0.000E+00	0.0000	271800052.834
27	10	0.100	4	0.000	271808355.606	0.000E+00	0.0000	271808355.606
28	10	1	4	0.000	271891383.318	0.000E+00	0.0000	271891383.318
29	10	10	4	0.000	272721660.437	0.000E+00	0.0000	272721660.437
30	10	100	4	0.000	281024431.628	0.000E+00	0.0000	281024431.628
31	100	0.001	4	0.000	2710237895.296	0.000E+00	0.0000	2710237895.296
32	100	0.010	4	0.000	2710238725.573	0.000E+00	0.0000	2710238725.573
33	100	0.100	4	0.000	2710247028.344	0.000E+00	0.0000	2710247028.344
34	100	1	4	0.000	2710330056.056	0.000E+00	0.0000	2710330056.056
35	100	10	4	0.000	2711160333.175	0.000E+00	0.0000	2711160333.175
36	100	100	4	0.016	2719463104.366	0.000E+00	0.0000	2719463104.366

Table 2 Results for 49 retailers and 49 candidate warehouse locations chosen from the 49-city data set (retailers in cities $\{1-49\}$, candidate warehouse locations in cities $\{1-49\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$, $\beta_{\text{inv}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 49-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 49-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	0.001	1	0.078	224371.213	2.910E−10	0.0000
2	0.001	0.010	1	0.078	227205.891	2.328E−10	0.0000
3	0.001	0.100	1	0.094	255552.669	2.619E−10	0.0000
4	0.001	1	1	0.078	539020.450	2.328E−10	0.0000
5	0.001	10	1	0.484	3373698.259	1.397E−09	0.0000
6	0.001	100	1	1.420	31720476.352	0.000E+00	0.0000
7	0.010	0.001	1	1.420	1142877.451	0.000E+00	0.0000
8	0.010	0.010	1	0.936	1145712.129	0.000E+00	0.0000
9	0.010	0.100	1	0.281	1174058.907	0.000E+00	0.0000
10	0.010	1	1	0.764	1457526.688	0.000E+00	0.0000
11	0.010	10	1	1.498	4292204.498	0.000E+00	0.0000
12	0.010	100	1	1.778	32638982.591	0.000E+00	0.0000
13	0.100	0.001	4	0.281	8699402.387	0.000E+00	0.0000
14	0.100	0.010	4	0.218	8702356.867	0.000E+00	0.0000
15	0.100	0.100	4	1.030	8731901.667	0.000E+00	0.0000
16	0.100	1	4	0.437	9027349.664	0.000E+00	0.0000
17	0.100	10	4	0.312	11981282.406	1.863E−09	0.0046
18	0.100	100	3	6.131	41456373.509	0.000E+00	0.1695
19	1	0.001	12	4.384	77858549.481	2.980E−08	0.0000
20	1	0.010	12	4.586	77861692.825	0.000E+00	0.0000
21	1	0.100	12	2.902	77893126.262	0.000E+00	0.0000
22	1	1	12	1.186	78207460.629	2.980E−08	0.0000
23	1	10	11	2.434	81336317.615	1.490E−08	0.0178
24	1	100	8	0.952	112593155.815	0.000E+00	0.1697
25	10	0.001	20	1.326	756654372.395	1.192E−07	0.0000
26	10	0.010	20	1.357	756657660.527	1.192E−07	0.0000
27	10	0.100	20	0.796	756690541.847	1.192E−07	0.0000
28	10	1	20	0.406	757019355.053	0.000E+00	0.0000
29	10	10	19	1.466	760337964.728	0.000E+00	−0.0040
30	10	100	15	0.702	793237736.173	2.384E−07	−0.0062
31	100	0.001	25	0.359	7528209253.929	9.537E−07	0.0000
32	100	0.010	25	4.306	7528212631.264	0.000E+00	0.0000
33	100	0.100	25	2.465	7528246404.605	0.000E+00	0.0000
34	100	1	25	0.359	7528584138.016	9.537E−07	0.0000
35	100	10	25	0.764	7531961472.129	0.000E+00	0.0000
36	100	100	25	18.018	7565734813.262	3.815E−06	0.0000

Table 3 Results for 88 retailers and 88 candidate warehouse locations chosen from the 88-city data set (retailers in cities $\{1-88\}$, candidate warehouse locations in cities $\{1-88\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$, $\beta_{\text{inv}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 88-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 88-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	0.001	1	9.391	312002.081	4.075E−10	0.0000
2	0.001	0.010	1	10.842	316751.106	3.492E−10	0.0000
3	0.001	0.100	1	7.675	364241.359	1.746E−10	0.0000
4	0.001	1	1	7.441	839143.891	2.328E−10	0.0000
5	0.001	10	1	7.067	5588169.212	1.863E−09	0.0000
6	0.001	100	1	52.541	53078422.423	1.490E−08	0.0000
7	0.010	0.001	2	10.842	1960400.948	2.328E−10	0.0000
8	0.010	0.010	2	9.064	1965201.525	4.657E−10	0.0000
9	0.010	0.100	2	4.586	2013207.293	2.328E−10	0.0000
10	0.010	1	2	2.465	2493264.972	0.000E+00	0.0000
11	0.010	10	1	6.037	7293438.913	2.794E−09	0.0055
12	0.010	100	1	4.727	54783692.124	0.000E+00	0.9417
13	0.100	0.001	8	120.058	15848529.133	2.025E+04	−0.0270
14	0.100	0.010	9	109.419	15849285.880	1.595E+04	0.0000
15	0.100	0.100	8	108.077	15903490.744	1.936E+04	−0.0247
16	0.100	1	9	109.419	16399879.504	7.646E+03	0.0155
17	0.100	10	6	33.353	21326184.292	7.451E−09	0.4908
18	0.100	100	3	41.075	70017159.835	1.490E−08	2.4252
19	1	0.001	18	74.537	141735536.003	2.737E+02	0.0000
20	1	0.010	18	49.718	141740765.059	0.000E+00	0.0000
21	1	0.100	18	26.863	141793055.611	0.000E+00	0.0000
22	1	1	18	11.591	142315961.138	2.980E−08	0.0000
23	1	10	17	22.495	147568926.917	0.000E+00	−0.0162
24	1	100	12	28.127	199923982.852	5.960E−08	−0.0442
25	10	0.001	37	23.166	1380140957.773	9.537E−07	0.0000
26	10	0.010	37	51.340	1380146507.479	7.153E−07	0.0000
27	10	0.100	37	7.114	1380202004.539	2.384E−07	0.0000
28	10	1	37	6.365	1380756975.131	0.000E+00	0.0000
29	10	10	34	15.616	1386331075.279	2.384E−07	−0.0018
30	10	100	22	16.037	1441817222.979	7.153E−07	−0.0009
31	100	0.001	47	71.542	13731509847.074	1.907E−06	0.0000
32	100	0.010	47	4.930	13731515513.398	0.000E+00	0.0000
33	100	0.100	47	19.048	13731572176.638	0.000E+00	0.0000
34	100	1	47	5.242	13732138809.040	0.000E+00	0.0000
35	100	10	47	75.286	13737805133.056	5.722E−06	0.0000
36	100	100	47	15.912	13794526343.888	0.000E+00	−0.0004

Table 4 Results for 150 retailers and 150 candidate warehouse locations chosen from the 150-city data set (retailers in cities $\{1-150\}$, candidate warehouse locations in cities $\{1-150\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$, $\beta_{\text{inv}} \in \{0.001, 0.01, 0.1, 1, 10, 100\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 150-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 150-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	0.001	1	459.205	491872.382	5.187E−05	0.0000
2	0.001	0.010	1	250.678	499184.009	1.746E−10	0.0000
3	0.001	0.100	1	299.272	572300.284	1.048E−09	0.0000
4	0.001	1	1	348.600	1303463.028	9.313E−10	0.0000
5	0.001	10	1	473.291	8615090.468	9.454E−01	0.0000
6	0.001	100	1	562.430	81731364.870	7.261E+01	0.0000
7	0.010	0.001	1	67.658	2661412.190	0.000E+00	0.0000
8	0.010	0.010	1	81.011	2668723.818	4.657E−10	0.0000
9	0.010	0.100	1	153.474	2741840.092	0.000E+00	0.0000
10	0.010	1	1	196.904	3473002.836	1.397E−09	0.0000
11	0.010	10	1	165.938	10784630.276	7.451E−09	0.0000
12	0.010	100	1	50.794	83900904.678	0.000E+00	0.0000
13	0.100	0.001	7	151.586	20339529.745	0.000E+00	0.0000
14	0.100	0.010	7	133.131	20347043.995	1.118E−08	0.0000
15	0.100	0.100	7	317.150	20422186.494	2.428E+00	0.0000
16	0.100	1	7	339.474	21173611.490	2.892E+02	0.0000
17	0.100	10	6	158.310	28740242.602	1.490E−08	−0.1823
18	0.100	100	3	160.463	103748227.856	4.470E−08	0.0792
19	1	0.001	20	218.620	179924475.758	2.086E−07	0.0000
20	1	0.010	20	186.967	179932325.925	3.278E−07	0.0000
21	1	0.100	20	255.467	180010827.588	3.874E−07	0.0000
22	1	1	20	222.582	180795844.222	5.960E−08	0.0000
23	1	10	18	199.712	188699514.211	2.980E−08	−0.0284
24	1	100	13	218.573	266773048.368	5.960E−08	0.1404
25	10	0.001	51	310.458	1733699031.838	1.082E−02	0.0000
26	10	0.010	51	150.432	1733707380.707	7.153E−07	0.0000
27	10	0.100	51	77.673	1733790869.406	7.153E−07	0.0000
28	10	1	51	335.075	1734625756.392	2.214E+01	0.0000
29	10	10	49	137.140	1743034093.294	4.768E−07	−0.0034
30	10	100	32	80.169	1828811509.177	0.000E+00	−0.1284
31	100	0.001	67	285.341	17208714121.291	2.289E−05	0.0000
32	100	0.010	67	275.935	17208722684.855	1.144E−05	0.0000
33	100	0.100	67	37.752	17208808320.494	0.000E+00	0.0000
34	100	1	67	243.580	17209664676.884	3.815E−06	0.0000
35	100	10	67	240.741	17218228240.787	7.629E−06	0.0000
36	100	100	59	239.727	17305227383.997	7.629E−06	−0.0079

joint optimization algorithm is not guaranteed to find the optimal solution to the subproblems $S_j(\lambda)$; if it does not solve all such subproblems to optimality, then the "lower bound" obtained by Lagrangian relaxation is not necessarily a true lower bound. In contrast, the sequential optimization approach uses Lagrangian relaxation to solve a transportation-location problem. In this case, the subproblems $S_j(\lambda)$ can be solved to optimality, and as a result a it is more likely that a true lower bound will be obtained. Thus negative VOI highlight a fundamental tradeoff between the benefits gained from integration versus the costs associated with using a Lagrangian-relaxation-based algorithm that is not guaranteed to solve the associated subproblems to optimality.

It should be mentioned that it is possible to modify the joint optimization algorithm **SolveMJIL** (Algorithm 2) so that negative VOI do not arise. The modification is simply to first obtain a solution using **SolveMJIL** (Algorithm 2), then obtain a solution using the sequential optimization algorithm, and finally to take whichever of these solutions gives the smallest value for the objective function (15) as the solution to the MJIL problem. Since the value of this solution cannot be larger than the value obtained by sequential optimization alone, the VOI cannot be negative. While the resulting algorithm would perform no worse than either the current **SolveMJIL** or the sequential optimization algorithm, and would yield non-negative VOI, there is no guarantee that the resulting solution would be optimal.

Based on the results in Tables 1–4, the region of the $(\beta_{\text{trn}}, \beta_{\text{inv}})$ parameter space in which there appears to be a clear benefit to integration is a subset of

$$\{(\beta_{\text{trn}}, \beta_{\text{inv}}) : \beta_{\text{trn}} \leq 1, \beta_{\text{inv}} \text{ large}\}.$$

We therefore decided to investigate this region further for each data set by varying the values of the transportation cost weighting factor β_{trn} and the inventory cost weighting factor β_{inv} such that

$$\beta_{\text{trn}} \in \{0.001, 0.01, 0.05, 0.1, 0.5, 1\} \quad \text{and} \quad \beta_{\text{inv}} \in \{100, 150, 225, 337.5, 506.25, 759.375\}.$$

These results, provided in Tables 5–8, reveal many positive VOI (along with some that are non-positive), demonstrating a clear benefit to integration over much of this region. The regions of the $(\beta_{\text{trn}}, \beta_{\text{inv}})$ parameter space in which there is a benefit to integration will likely be problem-dependent.

Another pattern that emerged is that the number of opened warehouses tended to increase as the transportation cost weighting factor β_{trn} was increased. This effect was particularly evident in Tables 1–4, which include larger values of β_{trn} than Tables 5–8. An explanation for this behavior is that as transportation costs increase, the fixed cost of opening and operating a warehouse becomes less of a barrier to opening a warehouse.

6 Conclusions and future research directions

In this paper we developed a Lagrangian relaxation-based heuristic for solving an integrated supply chain model that simultaneously considers facility location decisions as well as inventory policies at warehouses and retailers. We observed that, while over much of the $(\beta_{\text{trn}}, \beta_{\text{inv}})$ parameter space there was no discernible benefit to integration for these particular datasets and parameter values, over specific regions of this space there was a clear benefit to integration. We further observed that the number of opened warehouses tended to increase as the transportation cost weighting factor β_{trn} was increased.

There are a number of ways in which these results can be extended:

Table 5 Results for 11 retailers and 5 candidate warehouse locations chosen from the 49-city data set (retailers in cities $\{1-11\}$, candidate warehouse locations in cities $\{4, 11, 24, 35, 47\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.05, 0.1, 0.5, 1\}$, $\beta_{\text{inv}} \in \{100, 150, 225, 337.5, 506.25, 759.375\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 49-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 49-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI	Optimal value
1	0.001	100.000	1	0.047	8562526.023	1.863E−09	0.0000	8562526.023
2	0.001	150.000	1	0.078	12735646.737	0.000E+00	0.0000	12735646.737
3	0.001	225.000	1	0.062	18995327.808	0.000E+00	0.0000	18995327.808
4	0.001	337.500	1	0.125	28384849.415	3.725E−09	0.0000	28384849.415
5	0.001	506.250	1	0.140	42469131.825	0.000E+00	0.0000	42469131.825
6	0.001	759.375	1	0.281	63595555.440	7.451E−09	0.0000	63595555.440
7	0.01	100.000	1	0.031	8884723.311	1.863E−09	0.0000	8884723.311
8	0.01	150.000	1	0.047	13057844.025	0.000E+00	0.0000	13057844.025
9	0.01	225.000	1	0.062	19317525.096	3.725E−09	0.0000	19317525.096
10	0.01	337.500	1	0.062	28707046.703	0.000E+00	0.0000	28707046.703
11	0.01	506.250	1	0.078	42791329.113	0.000E+00	0.0000	42791329.113
12	0.01	759.375	1	0.078	63917752.727	1.490E−08	0.0000	63917752.727
13	0.05	100.000	1	0.047	10267650.841	0.000E+00	3.1582	10267650.841
14	0.05	150.000	1	0.047	14440771.555	1.863E−09	3.6079	14440771.555
15	0.05	225.000	1	0.218	20700452.626	0.000E+00	3.9425	20700452.626
16	0.05	337.500	1	0.140	30089974.233	3.725E−09	4.1834	30089974.233
17	0.05	506.250	1	0.359	44174256.643	2.025E+02	4.3527	44174256.643
18	0.05	759.375	2	0.484	68441978.742	3.143E+06	−0.3251	65300680.258
19	0.10	100.000	1	0.016	11904627.672	1.863E−09	1.4174	11904627.672
20	0.1	150.000	1	0.047	16077748.386	3.725E−09	2.2732	16077748.386
21	0.1	225.000	1	0.031	22337429.457	0.000E+00	2.9573	22337429.457
22	0.1	337.500	1	0.031	31726951.064	0.000E+00	3.4773	31726951.064
23	0.1	506.250	1	0.078	45811233.474	0.000E+00	3.8577	45811233.474
24	0.1	759.375	1	0.047	66937657.089	0.000E+00	4.1281	66937657.089
25	0.5	100.000	3	0.000	23337424.895	3.725E−09	0.0000	23337424.895
26	0.5	150.000	3	0.000	27866301.308	0.000E+00	0.0000	27866301.308
27	0.5	225.000	3	0.062	34659615.928	0.000E+00	0.0000	34659615.928
28	0.5	337.500	2	0.109	44630208.831	0.000E+00	0.4915	44630208.831
29	0.5	506.250	1	0.000	58805778.450	0.000E+00	2.2596	58805778.450
30	0.5	759.375	1	0.000	79932202.065	1.490E−08	3.9155	79932202.065
31	1	100.000	3	0.000	36948346.963	0.000E+00	0.0000	36948346.963
32	1	150.000	3	0.000	41477223.376	0.000E+00	0.0000	41477223.376
33	1	225.000	3	0.000	48270537.996	0.000E+00	0.0000	48270537.996
34	1	337.500	3	0.000	58460509.926	0.000E+00	0.0000	58460509.926
35	1	506.250	3	0.016	73745467.820	0.000E+00	0.0000	73745467.820
36	1	759.375	1	0.016	96175383.285	0.000E+00	0.5173	96142080.591

Table 6 Results for 49 retailers and 49 candidate warehouse locations chosen from the 49-city data set (retailers in cities {1 − 49}, candidate warehouse locations in cities {1 − 49}), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.05, 0.1, 0.5, 1\}$, $\beta_{\text{inv}} \in \{100, 150, 225, 337.5, 506.25, 759.375\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 49-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 49-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	100.000	1	1.388	31720476.352	0.000E+00	0.0000
2	0.001	150.000	1	1.685	47468686.404	1.490E−08	0.0000
3	0.001	225.000	1	2.402	71091001.482	0.000E+00	0.0000
4	0.001	337.500	1	3.557	106524474.099	0.000E+00	0.0000
5	0.001	506.250	1	6.131	159674683.024	0.000E+00	0.0000
6	0.001	759.375	1	8.003	239399996.411	5.960E−08	0.0000
7	0.01	100.000	1	1.794	32638982.591	0.000E+00	0.0000
8	0.01	150.000	1	6.287	48387192.643	1.490E−08	0.0000
9	0.01	225.000	1	0.967	72009507.720	0.000E+00	0.0000
10	0.01	337.500	1	1.061	107442980.337	1.490E−08	0.0000
11	0.01	506.250	1	2.262	160593189.262	0.000E+00	0.0000
12	0.01	759.375	1	2.496	240318502.649	5.960E−08	0.0000
13	0.05	100.000	1	0.234	36721232.540	1.490E−08	2.0264
14	0.05	150.000	1	0.780	52469442.592	7.451E−09	2.6867
15	0.05	225.000	1	0.889	76091757.669	1.490E−08	3.1646
16	0.05	337.500	1	1.092	111525230.286	0.000E+00	3.5019
17	0.05	506.250	1	1.076	164675439.211	0.000E+00	3.7357
18	0.05	759.375	1	0.920	244400752.598	0.000E+00	3.8958
19	0.10	100.000	3	5.803	41456373.509	0.000E+00	0.1695
20	0.1	150.000	2	4.789	57513461.829	2.980E−08	0.7423
21	0.1	225.000	1	0.421	81194570.105	0.000E+00	1.6830
22	0.1	337.500	1	1.248	116628042.722	0.000E+00	2.4557
23	0.1	506.250	1	1.045	169778251.647	0.000E+00	3.0100
24	0.1	759.375	1	1.061	249503565.034	2.980E−08	3.3987
25	0.5	100.000	6	2.761	73766781.604	2.980E−08	0.7125
26	0.5	150.000	4	0.515	90420915.658	0.000E+00	1.2179
27	0.5	225.000	4	0.998	115036293.380	1.490E−08	2.0258
28	0.5	337.500	4	3.557	151954798.213	0.000E+00	2.7501
29	0.5	506.250	4	4.633	207303994.956	0.000E+00	3.3671
30	0.5	759.375	2	5.039	289400438.709	0.000E+00	4.1842
31	1	100.000	8	0.998	112593155.815	0.000E+00	0.1697
32	1	150.000	8	4.914	129619641.268	0.000E+00	0.4842
33	1	225.000	6	2.839	154901908.397	0.000E+00	0.9941
34	1	337.500	4	2.153	192573020.176	0.000E+00	1.6412
35	1	506.250	4	4.087	247957620.051	0.000E+00	2.7076
36	1	759.375	4	4.774	331011123.971	5.960E−08	3.6454

Table 7 Results for 88 retailers and 88 candidate warehouse locations chosen from the 88-city data set (retailers in cities $\{1-88\}$, candidate warehouse locations in cities $\{1-88\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.05, 0.1, 0.5, 1\}$, $\beta_{\text{inv}} \in \{100, 150, 225, 337.5, 506.25, 759.375\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 88-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 88-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	100.000	1	52.526	53078422.423	1.490E−08	0.0000
2	0.001	150.000	1	16.474	79461896.430	2.980E−08	0.0000
3	0.001	225.000	1	17.831	119037107.439	1.490E−08	0.0000
4	0.001	337.500	1	15.663	178399923.953	2.980E−08	0.0000
5	0.001	506.250	1	110.714	267444148.723	0.000E+00	0.0000
6	0.001	759.375	1	15.787	401010485.879	2.384E−07	0.0000
7	0.01	100.000	1	4.774	54783692.124	0.000E+00	0.9417
8	0.01	150.000	2	91.183	80963661.514	4.172E−07	1.2423
9	0.01	225.000	1	26.333	120742377.140	4.470E−08	1.0203
10	0.01	337.500	2	89.966	179972333.836	6.557E−07	1.1164
11	0.01	506.250	1	13.010	269149418.424	1.192E−07	1.0563
12	0.01	759.375	2	146.423	402691359.709	2.236E+05	1.0721
13	0.05	100.000	2	31.278	62094610.315	0.000E+00	1.9244
14	0.05	150.000	2	47.237	88455382.626	1.788E−07	2.5742
15	0.05	225.000	1	24.399	128320664.905	2.980E−08	2.7868
16	0.05	337.500	2	33.244	187503021.556	2.980E−08	3.2747
17	0.05	506.250	2	13.572	276615024.033	0.000E+00	3.4878
18	0.05	759.375	2	30.904	410051151.535	2.384E−07	3.6925
19	0.10	100.000	3	41.262	70017159.835	1.490E−08	2.4252
20	0.1	150.000	4	29.297	97372745.387	2.980E−08	2.3397
21	0.1	225.000	2	29.469	137023470.036	0.000E+00	3.3068
22	0.1	337.500	1	9.454	197081513.297	2.980E−08	3.7185
23	0.1	506.250	1	5.210	286125738.068	0.000E+00	4.3923
24	0.1	759.375	2	26.629	419348533.378	0.000E+00	4.9528
25	0.5	100.000	8	18.237	129231346.394	0.000E+00	0.4719
26	0.5	150.000	6	17.269	157117644.737	5.960E−08	0.9119
27	0.5	225.000	9	19.157	199875015.158	0.000E+00	0.8702
28	0.5	337.500	5	38.111	260300056.607	0.000E+00	2.2704
29	0.5	506.250	5	39.187	352190293.579	5.960E−08	3.0987
30	0.5	759.375	3	53.618	487216376.367	0.000E+00	4.3569
31	1	100.000	12	28.813	199923982.852	5.960E−08	−0.0442
32	1	150.000	10	16.879	229956681.455	8.941E−08	−0.4657
33	1	225.000	9	33.727	270457669.429	0.000E+00	0.7408
34	1	337.500	6	80.543	332832804.368	1.490E−06	1.4998
35	1	506.250	4	22.355	428439003.124	0.000E+00	1.7343
36	1	759.375	5	27.051	565073529.393	0.000E+00	3.1612

Table 8 Results for 150 retailers and 150 candidate warehouse locations chosen from the 150-city data set (retailers in cities $\{1-150\}$, candidate warehouse locations in cities $\{1-150\}$), for $\beta_{\text{trn}} \in \{0.001, 0.01, 0.05, 0.1, 0.5, 1\}$, $\beta_{\text{inv}} \in \{100, 150, 225, 337.5, 506.25, 759.375\}$. The plant is at a location near Atlanta, Georgia, at longitude 84 and latitude 33. Per-unit shipping costs between the plant and warehouses were obtained by multiplying the great-circle distances between the locations by .04, and per unit shipping costs between the warehouses and the retailers were obtained by multiplying the great-circle distances between the locations by .05. Demand at a retailer was obtained by multiplying the 5th *column* in the 150-city dataset by .0001. The fixed order cost at a warehouse was obtained by multiplying the 6th *column* in the 150-city dataset by 2.5. For each retailer, the unit inventory cost and the fixed order cost were respectively \$4.30 and \$2.10, while for each warehouse, the unit inventory cost and the fixed order cost were respectively \$2.60 and \$1.80. The base planning period was taken as 365 days. DCs = Number of open warehouses, Gap = BestUpperBound − BestLowerBound, VOI = Value of Integration

No.	β_{trn}	β_{inv}	DCs	Time (sec.)	Obtained value	Gap	VOI
1	0.001	100.000	1	564.708	81731364.870	7.261E+01	0.0000
2	0.001	150.000	1	268.696	122351517.315	1.490E−08	0.0000
3	0.001	225.000	1	288.025	183281745.983	2.980E−08	0.0000
4	0.001	337.500	1	209.541	274677088.985	5.960E−08	0.0000
5	0.001	506.250	1	151.212	411770103.488	5.960E−08	0.0000
6	0.001	759.375	2	454.603	616026540.447	8.225E−06	0.2245
7	0.01	100.000	1	50.544	83900904.678	0.000E+00	0.0000
8	0.01	150.000	1	64.896	124521057.123	1.490E−08	0.0000
9	0.01	225.000	2	107.032	185404599.316	5.960E−08	0.0252
10	0.01	337.500	1	149.480	276846628.793	1.192E−07	0.0000
11	0.01	506.250	2	149.667	413545996.317	1.192E−07	0.0952
12	0.01	759.375	1	217.263	619579165.052	0.000E+00	0.0000
13	0.05	100.000	2	167.779	93096591.526	5.960E−08	0.5399
14	0.05	150.000	2	90.933	134005590.661	2.980E−08	0.7209
15	0.05	225.000	1	89.155	195093684.939	5.960E−08	0.9927
16	0.05	337.500	2	124.317	285922148.012	0.000E+00	1.4677
17	0.05	506.250	2	170.493	422711633.832	0.000E+00	1.6652
18	0.05	759.375	2	213.004	627813675.457	2.384E−07	1.8135
19	0.10	100.000	3	160.151	103748227.856	4.470E−08	0.0792
20	0.1	150.000	3	175.002	144271633.002	8.941E−08	0.9042
21	0.1	225.000	3	126.610	206718960.827	1.192E−07	0.7140
22	0.1	337.500	2	101.572	297902111.214	1.192E−07	1.4169
23	0.1	506.250	2	146.797	434992899.268	1.788E−07	1.8442
24	0.1	759.375	2	192.599	640473088.339	7.153E−07	2.1672
25	0.5	100.000	9	172.085	178028959.356	5.960E−08	0.1314
26	0.5	150.000	7	138.763	221314308.622	0.000E+00	0.0040
27	0.5	225.000	5	195.204	283462239.451	5.960E−08	0.8648
28	0.5	337.500	6	131.056	379562720.758	5.960E−08	0.8527
29	0.5	506.250	4	186.858	517284660.791	2.384E−07	2.0962
30	0.5	759.375	4	143.225	725642613.659	0.000E+00	2.8221
31	1	100.000	13	218.916	266773048.368	5.960E−08	0.1404
32	1	150.000	16	180.212	312670947.200	1.192E−07	−0.6113
33	1	225.000	9	161.227	374386660.913	0.000E+00	0.4784
34	1	337.500	10	92.883	473053801.682	1.788E−07	0.2645
35	1	506.250	7	90.605	615337768.698	0.000E+00	1.0007
36	1	759.375	5	138.732	822957880.079	2.384E−07	2.3481

1. The model, which assumes the movement and storage of a single product, can be extended to consider multiple products.
2. Capacity constraints at the warehouses can be introduced. Stochastic capacity constraints can also be considered.
3. The single sourcing restriction can be relaxed to allow a single retailer to be supplied by more than one warehouse, which may be advantageous in capacitated models or in models with multiple products.
4. The assumption that lead times are negligible can be relaxed, and can be generalized to the case of stochastic lead times.
5. The inventory policy can be extended to include consideration of safety stock.
6. The model can be extended to allow lateral shipments between warehouses, which can result in substantial cost savings when the warehouses are owned by the same firm.
7. The assumption of direct shipments from warehouses to retailers can be relaxed, thus allowing routing decisions to be incorporated into the model, resulting in a model that combines location, inventory, and routing decisions.
8. The algorithm used to optimize the model can be improved.

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